Two-Scale Generalized Finite Element Methods for Three-Dimensional Cohesive Propagating Fractures

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Simulation of Reflective Crack Growth in Pavements

- Cracks and joints in a pavement with AC overlay “reflect” up to the surface, propagating through overlay

[Pictures by W. G. Buttlar]
Simulation of Reflective Crack Growth in Pavements

- Reflective crack testing at FAA – NAPTF – Simulation and life estimate

Computational challenges
- Highly localized non-linear 3-D effects around crack front
- RC surface change in size by orders of magnitude
- Fatigue cracking with thousands of cycles
- Crack channeling and coalescence

Goals and Strategy
- Simulate propagating cohesive fractures using coarse meshes
- Minimize non-linear iterations at problem scale
- A two-scale Generalized/eXtended FEM (G/XFEM) for 3-D cohesive fractures
Outline

- Motivation
- Generalized/eXtended finite element methods: Basic ideas
- Bridging scales with G/XFEM: Global-local enrichments for localized non-linearities
  - Example
- Alternative strategy for long cracks
  - Example
- Conclusions
Generalized/eXtended Finite Element Method

- GFEM is a Galerkin method with special test/trial space given by

\[ S_{GFEM} = S_{FEM} + S_{ENR} \]

Low order FEM space  Enrichment space with functions related to the given problem

\[ S_{FEM} = \left\{ \sum_{\alpha \in I_h} \hat{u}_\alpha \varphi_\alpha(x), \quad \hat{u}_\alpha \in \mathbb{R}^3 \right\} \]

\[ S_{ENR} = \left\{ \sum_{\alpha \in I^e_h \subset I_h} \sum_{i=1}^{m_\alpha} \tilde{u}_{\alpha i} \varphi_\alpha(x) L_{\alpha i}(x), \quad \tilde{u}_{\alpha i} \in \mathbb{R}^3 \right\} \]

\[ L_{\alpha i} \in \chi_\alpha(\omega_\alpha) \]

Enrichment function  Patch space
Generalized/eXtended Finite Element Method

\[ u^{hp}(x) = \sum_{\alpha \in I_h} \hat{u}_\alpha \varphi_\alpha(x) + \sum_{\alpha \in I^x_h \subset I_h} \sum_{i=1}^{m_\alpha} \tilde{u}_{\alpha i} \varphi_\alpha(x) L_{\alpha i}(x), \quad \hat{u}_\alpha, \tilde{u}_{\alpha i} \in \mathbb{R}^3 \]

\[ \phi_{\alpha i}(x) = \varphi_\alpha(x) L_{\alpha i}(x) \]

- Allows construction of shape functions incorporating a-priori knowledge about solution

[Oden, Duarte & Zienkiewicz, 1996]
Bridging Scales with Global-Local Enrichment Functions*

- Enrichment functions computed from solution of local boundary value problems: **Global-Local enrichment functions**

  ![Diagram of enrichment functions]

  - **Idea:** *Use available numerical solution at a simulation step to build shape functions for next step* (quasi-static, transient, non-linear, etc.)
  - Enrichment functions are produced numerically on-the-fly through a global-local analysis
  - Use a *coarse* mesh enriched with Global-Local (GL) functions
  - GFEM\textsubscript{gl} = GFEM with global-local enrichments

* [Duarte et al. 2005]
Global-Local Enrichments for Problems with Localized Non-Linearities*

- **Model Problem:** Simulation of propagating cracks using cohesive fracture models

\[ \begin{align*}
\int_{\Omega_G} \nabla^s (\delta u) : \sigma (u) \, dV + \int_{\Gamma_{coh}^{\text{t}}} \delta [u] \cdot t^{\text{coh}} ([u]) \, dS + \eta \int_{\Gamma_{G}^{u}} \delta u \cdot u \, dS \\
= \int_{\Omega_G} \delta u \cdot b \, dV + \int_{\Gamma_{G}^{t}} \delta u \cdot \bar{t} \, dS + \eta \int_{\Gamma_{G}^{u}} \delta u \cdot \bar{u} \, dS
\end{align*} \]

*[Jongheon Kim and C.A. Duarte, ijmne, 2015]*
Global-Local Enrichments for Problems with Localized Non-Linearities

- Three-Point Bending Beam

- Typical FEM discretization [Park et al. 2008]

Goals:
- Solve problem on a coarse global mesh.
- Minimize number of non-linear iterations at global (problem) scale.
Global-Local Enrichments for Problems with Localized Non-Linearities

Let \( u^n_G \in \mathcal{S}^n_G(\Omega) \) be a GFEM approximation of global problem at load step \( n \). Global-local enrichments are used in the definition of \( \mathcal{S}^n_G(\Omega) \).

Compute a cheap estimate of global solution at next load step.

Define \( u^{n+1}_{G,0} = \frac{n+1}{n} u^n_G \)

\( u^{n+1}_{G,0} \) is used to prescribe boundary conditions for a non-linear local problem as defined next.
Global-Local Enrichments for Problems with Localized Non-Linearities

- Solve following non-linear *local* problem at load step $n+1$ using, e.g., GFEM

Find $\mathbf{u}^{n+1}_L \in \mathbb{S}^{n+1}_L(\Omega_L)$ such that, $\forall \, \delta \mathbf{u}^{n+1}_L \in \mathbb{S}^{n+1}_L(\Omega_L)$

$$
\int_{\Omega_L} \nabla^s (\delta \mathbf{u}^{n+1}_L) : \mathbf{\sigma} (\mathbf{u}^{n+1}_L) \, dV + \int_{\Gamma_{\text{coh}}} \delta [\mathbf{u}^{n+1}_L] \cdot t_{\text{coh}}^{L} ([\mathbf{u}^{n+1}_L]) \, dS + \eta \int_{\Gamma_L \cap \Gamma_G^u} \delta \mathbf{u}^{n+1}_L \cdot \mathbf{u}^{n+1}_L \, dS \\
+ \kappa \int_{\Gamma_L \setminus (\Gamma_L \cap (\Gamma_G^u \cup \Gamma_G^t))} \delta \mathbf{u}^{n+1}_L \cdot \mathbf{u}^{n+1}_L \, dS = \int_{\Omega_L} \delta \mathbf{u}^{n+1}_L \cdot \mathbf{b} \, dV + \int_{\Gamma_L \cap \Gamma_G^t} \delta \mathbf{u}^{n+1}_L \cdot \mathbf{t} \, dS \\
+ \eta \int_{\Gamma_L \cap \Gamma_G^u} \delta \mathbf{u}^{n+1}_L \cdot \bar{\mathbf{u}} \, dS + \eta \int_{\Gamma_L \setminus (\Gamma_L \cap (\Gamma_G^u \cup \Gamma_G^t))} \delta \mathbf{u}^{n+1}_L \cdot \mathbf{u}^{n+1}_{G,0} \, dS
$$
Global-Local Enrichments for Problems with Localized Non-Linearities

- Global space enriched with non-linear local solution

\[ \phi_{\alpha}^{n+1}(x) = \varphi_{\alpha}(x)u_{L}^{n+1}(x) \]

\[ u_{G}^{n+1}(x) \in S_{G}^{n+1}(\Omega_{G}) = S_{G}^{\text{FEM}} + \{ \varphi_{\alpha}u_{\alpha}^{\text{gl},n+1}, \alpha \in I^{\text{gl}} \} \]

where \( u_{\alpha}^{\text{gl},n+1}(x) = \left\{ \begin{array}{l} u_{\alpha}u_{L}^{n+1,<0>}(x) \\ v_{\alpha}u_{L}^{n+1,<1>}(x) \\ w_{\alpha}u_{L}^{n+1,<2>}(x) \end{array} \right\} \), \( u_{\alpha}, v_{\alpha}, w_{\alpha} \in \mathbb{R} \)

- Discretization spaces updated on-the-fly with global-local enrichment functions
Global-Local Enrichments for Problems with Localized Non-Linearities

- On-the-fly updating of global-local enrichment functions during the non-linear iterative solution process
Global-Local Enrichments for Problems with Localized Non-Linearities

• Global space at load step $n+1$

\[
\mathbf{u}_{G}^{n+1}(\mathbf{x}) \in S_{G}^{n+1}(\Omega_{G}) = S_{G}^{\text{FEM}} + S_{G}^{\text{ENR},n+1}
\]

where \( S_{G}^{\text{ENR},n+1} = \{ \varphi_{\alpha} \mathbf{u}_{\alpha}^{gl,n+1}, \ \alpha \in I^{gl} \} \)

• Discretization spaces updated on-the-fly with global-local enrichment functions
• Enrichment space is *load dependent*
• Dimension of global space does not change but its basis functions do:

Use of vector with global DOFs computed at previous load step is *not* a robust choice for the initialization of the Newton-Rhapson non-linear iterations at this load step

\( \mathbf{d}_{G}^{n} \) and \( \mathbf{d}_{G}^{n+1} \) represent coefficients of different sets of GFEM shape functions

Classical strategy: Map \( \mathbf{u}_{G}^{n} \in S_{G}^{n}(\Omega_{G}) \) into \( S_{G}^{n+1}(\Omega_{G}) \)
Global-Local Enrichments for Problems with Localized Non-Linearities

- **Alternative:** Linear global problem at load step $n+1$

Find $\mathbf{u}^{n+1}_G \in \mathbb{S}^{n+1}_G(\Omega_G)$ such that, $\forall \delta \mathbf{u}^{n+1}_G \in \mathbb{S}^{n+1}_G(\Omega_G)$

$$
\int_{\Omega_G} \nabla^s (\delta \mathbf{u}^{n+1}_G) : \sigma (\mathbf{u}^{n+1}_G) \ dV + \int_{\Gamma_{\text{coh}}} \delta [\mathbf{u}^{n+1}_G] \cdot C^{m+1}_D (\mathbf{u}^{n+1}_L) [\mathbf{u}^{n+1}_G] \ dS + \eta \int_{\Gamma_{\text{ub}}} \delta \mathbf{u}^{n+1}_G \cdot \mathbf{u}^{n+1}_G \ dS
$$

$$
= \int_{\Omega_G} \delta \mathbf{u}^{n+1}_G \cdot \mathbf{b} \ dV + \int_{\Gamma_G} \delta \mathbf{u}^{n+1}_G \cdot \mathbf{t} \ dS + \eta \int_{\Gamma_{\text{ub}}} \delta \mathbf{u}^{n+1}_G \cdot \bar{\mathbf{u}} \ dS
$$

The cohesive secant matrix is given by

$$
C^{m+1}_D = \begin{bmatrix}
C^{m+1,<m_0>}_D & 0 & 0 \\
0 & C^{m+1,<m_1>}_D & 0 \\
0 & 0 & C^{m+1,<m_2>}_D
\end{bmatrix}
$$

where

$$
C^{m+1,<m_t>}_D = \frac{t^{<m_t>}}{t_{\text{coh}}} (\left[ \mathbf{u}^{n+1}_L \right])^{<m_t>}, \quad t = 0, 1, 2
$$
Global-Local Enrichments for Problems with Localized Non-Linearities

- A linear global problem with secant stiffness provided by local problem is solved at every load step.
- Non-linearities are handled at the local problem.
- BCs for local problem can be improved through global-local iterations.
Global-Local Enrichments for Problems with Localized Non-Linearities

Experiments by [Roesler et al., 2007], 2-D FEM results by [Park et al. 2008]

2-D FEM, $hp$-GFEM, and GFEM$^{gl}$ meshes
Control of Non-Linear Residue of Global Solution

- Non-linear residue of global solution computed with secant stiffness

- Residue is maximum at load step 37 while limit point is reached at load step 28
- One Newton-Raphason iteration reduces residue to $\sim 10^{-11}$ at all load steps
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  - Example
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Alternative Strategy for Long Cracks*

Simulation of cohesive fracture propagation at a material interface

Classical strategies
• Fixed mesh over entire expected crack path
• Adaptive remeshing and mapping of solution between meshes

*[J. Kim, A. Simone and C.A. Duarte, ijmne, 2016]*
Alternative Strategy for Long Cracks

• Split simulation into sub-simulations [Shabir, 2012]

- For each sub-simulation,
  - Single refinement window is defined around cohesive zone
  - Locally refined mesh is used ONLY inside refinement windows
  - Solve using Newton-Rhapson method with the incremental strategy
Alternative Strategy for Long Cracks

End of sub-simulation

- refinement window
- potential crack surface

mapping of non-linear solutions?

direction of crack propagation

Start of next sub-simulation

How to start Newton-Rhapson iteration at next sub-simulation?
3-D crack surface representation

High-fidelity explicit representation of crack surfaces [Duarte et al., 2001, 2009]

Geometrical crack surface is independent of 3-D computational GFEM mesh:
  - Can be used to transfer information between sub-simulation meshes
Initialization of Newton-Rhapson algorithm

Non-linear problem at the load step $k$ and at the beginning of sub-simulation $s$

Find $u^k_s \in S^s(\Omega) \subset H^1(\Omega)$ such that for all $\delta u^k_s \in S^s(\Omega)$

$$\int_{\Omega} \nabla^s(\delta u^k_s) : \sigma(u^k_s) \, dV + \int_{\Gamma_{coh}} \delta [u^k_s] \{ t_{coh}(\|u^k_s\|) \} \, dS + \eta \int_{\Gamma^u} \delta u^k_s \cdot u^k_s \, dS$$

$$= \int_{\Omega} \delta u^k_s \cdot b^k \, dV + \int_{\Gamma^t} \delta u^k_s \cdot \bar{t}^k \, dS + \eta \int_{\Gamma^u} \delta u^k_s \cdot \bar{u}^k \, dS.$$

Initial guess from the solution of a linear problem

$$\int_{\Omega} \nabla^s(\delta u^k_s) : \sigma(u^k_s) \, dV + \int_{\Gamma_{coh}} \delta [u^k_s] \{ C_{s,k}(\|u^k_{s-1}\|)[u^k_s] \} \, dS + \eta \int_{\Gamma^u} \delta u^k_s \cdot u^k_s \, dS$$

$$= \int_{\Omega} \delta u^k_s \cdot b^k \, dV + \int_{\Gamma^t} \delta u^k_s \cdot \bar{t}^k \, dS + \eta \int_{\Gamma^u} \delta u^k_s \cdot \bar{u}^k \, dS.$$
Initialization of Newton-Rhapson algorithm: Secant stiffness

Linear problem at the load step $k$ and at the beginning of sub-simulation $s$

\[
\int_{\Omega} \nabla^s (\delta u^k_s) : \sigma(u^k_s) \, dV + \int_{\Gamma_{coh}} \delta [u^k_s] \cdot C_{D}^{s,k} ([u^k_{s-1}]) [u^k_s] \, dS + \eta \int_{\Gamma_u} \delta u^k_s \cdot u^k_s \, dS
\]

\[
= \int_{\Omega} \delta u^k_s \cdot b^k \, dV + \int_{\Gamma_t} \delta u^k_s \cdot t^k \, dS + \eta \int_{\Gamma_u} \delta u^k_s \cdot \bar{u}^k \, dS.
\]
Initialization of Newton-Rhapson algorithm: Data transfer

*Geometrical* and *computational* crack surfaces [Gupta & Duarte, 2014]

- Geometrical crack surface is used to describe crack geometry
- Computational crack surface is used to integrate on the crack surface

**Strategy:** Store $u_{s-1}^k$ on the closest facets of geometrical crack surface to each integration point of the computational crack surface
Delamination test

• $\overline{u}_1 = \overline{u}_2$: mode I, $\overline{u}_1 = 2\overline{u}_2$: mixed mode

• Isotropic linear elasticity in the bulk

• Linear intrinsic cohesive model:

$$t^{(i)}_{coh} = \begin{cases} \max [0, f_i + k_i (\|u\|^{(i)} - \delta_{ini,i})] & \text{if } \|u\|^{(i)} < \delta_{ini,i} \\ \text{if } \|a\|^{(i)} \geq \delta_{ini,i} & \text{if } i = N, M \end{cases}$$
Delamination test: Reference solution

Fixed mesh

Mode I

Mixed mode

Syy

-50 -40 -20 0 20 40 50
Delamination test: Adaptive meshes

sub-simulation 1

sub-simulation 2

sub-simulation 3

sub-simulation 4

sub-simulation 5

sub-simulation 6

sub-simulation 7
Delamination test: Recovered stress fields

Stresses at the end of each sub-simulation

Stresses recovered at the next sub-simulation
Delamination test: Results (mode I)

Load-displacement curves

<table>
<thead>
<tr>
<th>Strategy</th>
<th>Time (s)</th>
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<tr>
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<td>50,627</td>
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Delamination test: Results (mixed mode)

Load-displacement curves

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<td>45,854</td>
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<td>Adaptive meshes</td>
<td>26,751</td>
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Simplified strategy: Adaptation to standard FEMs

- At the beginning of each sub-simulation,
  - Solve from scratch to recover a non-linear solution at the current load level
  - No data transfer is needed between meshes
  - Can be used with commercial FEM software
Simplified strategy: Abaqus results

Stresses at the end of each sub-simulation

Stresses recovered at the next sub-simulation
Simplified strategy: Abaqus results

3-D fixed mesh

2-D fixed mesh

Load-displacement curves

CPU time

<table>
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Abaqus files available from: http://simonelab.tudelft.nl/index_gfemsequential.html#gfemsequential
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